MA26500: EXAM I **February 21, 2017**

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NAME: ____________________ **PUID:**____________________

Section Number: ____________________ **Class Time:**____________________

(1) No calculators are allowed.

(2) No portable electronic devices.

(3) There are 10 problems. Each problem is worth 10 points.

(4) The score is accumulative and the maximum is 100.

1 B

2 D

3 A

4 D

5 B

6 C

7 E

8 D

9 A

10 D

1. If $A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 4 \\ 3 & -2 & 0 \end{bmatrix}$. What is the $(1, 2)$ entry A^{-1}?

A. $\frac{12}{55}$

B. $\frac{6}{55}$

C. $\frac{2}{11}$

D. $-\frac{1}{11}$

E. 5

$$A_{21} = -\begin{vmatrix} 0 & -3 \\ -2 & 0 \end{vmatrix} = 6$$

$$|A| = \begin{vmatrix} 2 & 0 & -3 \\ -1 & 5 & 4 \\ 3 & -2 & 0 \end{vmatrix} = -6 + 45 + 16 = 55$$

The $(1, 2)$ entry of A^{-1} is $\dfrac{6}{55}$

2. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, then $AB - BA$ equals

A. $\begin{bmatrix} 5 & 0 \\ -10 & 5 \end{bmatrix}$

B. $\begin{bmatrix} -5 & 0 \\ -10 & 5 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 0 \\ 10 & -5 \end{bmatrix}$

D. $\begin{bmatrix} 5 & -10 \\ 0 & -5 \end{bmatrix}$

E. $\begin{bmatrix} -5 & 10 \\ 0 & 5 \end{bmatrix}$

answer: D

$$AB = \begin{bmatrix} 7 & -1 \\ 3 & -4 \end{bmatrix} \qquad BA = \begin{bmatrix} 2 & 9 \\ 3 & 1 \end{bmatrix}$$

$$AB - BA = \begin{bmatrix} 5 & -10 \\ 0 & -5 \end{bmatrix}$$

3. A is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector. $A\mathbf{x} = \mathbf{b}$ has a unique solution. Consider the following statements:

(i) $m < n$

(ii) $n \leq m$

(iii) the rank of $A < n$

(iv) the rank of $A = m$

Which **must** be true?

A. only (ii)

B. only (iv)

C. only (i) and (iii)

D. only (iii)

E. None of the statements has to be true.

Answer: A

$Ax = b$ has a unique solution

$$\Downarrow$$

$$\text{rank}(A) = n$$

So (iii) is WRONG.

$$n = \text{rank}(A) \leq \#\text{ of rows} = m$$

So (ii) is CORRECT

Counter example for (i)

$$\begin{cases} x = 1 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Counter example for (iv)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

4. Assume that two 3×3 matrices A and B are row equivalent. Which of the following statements is **wrong**?

A. A and B have the same rank.

B. A and B have the same reduced row echelon form.

C. The homogeneous systems $Ax = 0$ and $Bx = 0$ have the same set of solutions.

D. For any $b \in \mathbb{R}^3$, the inhomogeneous systems $Ax = b$ and $Bx = b$ have the same set of solutions.

E. For any 3×3 matrix C, the two matrices AC and BC are also row equivalent.

answer: D

Counter example for D :

$$A = I_3 \sim B = 2 I_3$$

$A \sim B \Leftrightarrow A = E_1 \cdots E_k B$ $\quad E_i$ are elementary matrices.

So $AC = E_1 \cdots E_k BC$

$$\Updownarrow$$

$$A \sim B$$

So E is CORRECT.

5. Determine which one of the following expressions is the general solution to the inhomogeneous system of equations

$$\begin{cases} 5x_1 - 6x_2 + x_3 &= 4 \\ 2x_1 - 3x_2 + x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 5 \end{cases}$$

A. $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

D. $s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

E. No solution.

answer: B

$$\left[\begin{array}{ccc|c} 5 & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{array}\right] \underset{\substack{A_{12}(-2) \\ A_{13}(-4)}}{\overset{A_{31}(-1)}{\sim}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 9 & -9 & 9 \end{array}\right]$$

$$\underset{A_{23}(-1)}{\overset{\substack{M_2(\frac{1}{3}) \\ M_3(\frac{1}{9})}}{\sim}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{A_{21}(3)}{\sim} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

$$\begin{bmatrix} x = \\ y = \\ z = \end{bmatrix} \begin{bmatrix} 2+s \\ 1+s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

6. Find all the value(s) of k such that $\begin{bmatrix} 1 \\ 2-k \\ -2 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} k \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ k+2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

A. $k = 1$

B. $k = 1, -3$

C. $k \neq 1, -3$

D. $k \neq 1$

E. $k = -3$

answer: C

$$\det \begin{bmatrix} k & 1 & 1 \\ 3 & k+2 & -1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{vmatrix} k & 1 & 1 \\ 3 & k+2 & -1 \\ 2-2k & 0 & 0 \end{vmatrix} = (2-2k) \begin{vmatrix} 1 & 1 \\ k+2 & -1 \end{vmatrix}$$

$$= 2(k-1)(k+3) = 0 \quad \Rightarrow \quad k = 1, -3.$$

When $k \neq 1, -3$ $\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 3 & k+2 & -1 & 2-k \\ 2 & 2 & 2 & -2 \end{array}\right]$ always has a unique sol.

When $k = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & 1 \\ 2 & 2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right] \text{ No sol.}$$

When $k = -3$

$$\left[\begin{array}{ccc|c} -3 & 1 & 1 & 1 \\ 3 & -1 & -1 & 5 \\ 2 & 2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 6 \\ 2 & 2 & 2 & -2 \end{array}\right] \text{ No sol.}$$

7. Determine all values of k so that $\{t^2 + kt + 1, t^2 - t + 1, 2t - k\}$ is **NOT** a spanning set of $\mathbb{P}_2 = \{\text{all polynomials of degree no more than } 2\}$.

A. $k \neq 1$

B. $k \neq 0, -1$

C. $k \neq 0$

D. $k = 0, 1$

E. $k = 0, -1$

answer: E

The given set is NOT a spanning set of P_2 if and only if

$$0 = \begin{vmatrix} 1 & 1 & 0 \\ k & -1 & 2 \\ 1 & 1 & -k \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ k & -1 & 2 \\ 0 & 0 & -k \end{vmatrix}$$

$$= -k \begin{vmatrix} 1 & 1 \\ k & -1 \end{vmatrix} = k(k+1)$$

$\Rightarrow k = 0, -1$

8. For what values of the constant k, does the linear system

$$AX = 0, \quad A = \begin{bmatrix} 1 & k & k & -k \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 1 & k \\ 0 & 10 & 1 & k^2 \end{bmatrix}$$

have infinitely many solutions?

A. no value of k

B. $k \neq 1, 3$

C. $k \neq 1, 2$

D. $k = 1, 3$

E. $k = 1, 2$

answer: D

$AX = 0$ has ∞ many sol $\Leftrightarrow |A| = 0$

$$\begin{vmatrix} 1 & k & k & -k \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 1 & k \\ 0 & 10 & 1 & k^2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 4 & 1 & k \\ 10 & 1 & k^2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & k-1 \\ 8 & 1 & k^2-1 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 2 & k-1 \\ 8 & k^2-1 \end{vmatrix} = -(2k^2 - 2 - 8k + 8)$$

$$= -2(k^2 - 4k + 3) = 0 \Rightarrow k = 1, 3$$

9. Which of the following sets S are subspaces of the given vector space V?

(i) $S = \{A \in V : A^T = -A\}$, $V = M_3(\mathbb{R}) = \{3 \times 3 \text{ matrices with real entries}\}$

(ii) $S = \{f(t) = at^2+bt+c : f(3) = 2\}$, $V = \mathbb{P}_2 = \{\text{all polynomials of degree no more than } 2\}$

(iii) $S = \{A \in V : Ax = 0 \text{ only has the trivial solution}\}$, $V = M_4(\mathbb{R}) = \{4\times 4 \text{ matrices with real entries}\}$

(iv) $S = \{(x, y) \in V : y = 2x + 1\}$, $V = \mathbb{R}^2$

(v) $S = \{(x, y, z) \in V : x^2 + 4y^2 = z\}$, $V = \mathbb{R}^3$

A. only (i)

B. only (i) and (iii)

C. only (i), (iii) and (iv)

D. only (ii) and (v)

E. All of the above.

answer: A

✓ (i) S is set of all skew-sym matrices of size 3×3

For any $A, B \in S$, $c \in \mathbb{R}$

$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$

$$(cA)^T = c\,A^T = c(-A) = -(cA)$$

(ii) zero function $\notin S$.

(iii) $A = I_2$, $B = -I_2 \in S$, but $A+B = 0 \notin S$.

(iv) $(0, 0) \notin S$

(v) $(1, 0, 1) \in S$, but $(2, 0, 2) \notin S$.

10. Let A and B be 2×2 matrices. Which of the following statements is **always** true?

A. $(A+B)^2 = A^2 + 2AB + B^2$

B. If $A^2 = 0$, then $A = 0$.

C. If A is invertible, then so is AB.

D. If A, B are both invertible, then so is AB.

E. $|A+B| = |A| + |B|$.

A: $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$ if $AB \neq BA$

B: Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $A^2 = 0$

C: Take $A = I_2$ $B = 0$. then $AB = 0$ singular

D: $|AB| = \underset{\neq 0}{|A|}\ \underset{\neq 0}{|B|} \neq 0 \iff AB$ invertible

E: Take $A = I_2$ $B = -I_2$

$|A+B| = 0 \neq |A| + |B| = 2$